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A New Class of Almost Controllable Graphs

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Abstract

A graph G is called a chain graph if it contains none of the graphs $\{2K_2, C_3, C_5\}$ as induced subgraphs. It is said to be almost controllable if it has exactly one non-main eigenvalue, that is, precisely one eigenvalue whose corresponding eigenspace is orthogonal to the all-ones vector. Within the class of chain graphs, we identify new families of graphs exhibiting this almost controllable property.

Keywords: chain graphs; main or non-main eigenvalues; almost controllable graphs.

1 Introduction

We consider a simple graph G, that is, a graph without loops or multiple edges, whose vertex set is $V(G) = \{v_1, \dots, v_n\}$ and which contains m edges. The adjacency matrix $A(G)$ is a $0-1$ matrix where the entry $a_{ij} = 1$ if v_i and v_j are adjacent, and 0 otherwise. The eigenvalues of G are defined as those of $A(G)$, i.e., the roots of the characteristic polynomial $\phi_G(\lambda) = \det(\lambda I_n - A(G))$. The multiset of eigenvalues is called the *spectrum* of G and is denoted by $\sigma(G)$. An eigenvalue μ is called *main* if its eigenspace $\mathcal{E}_{G,\mu}$ is not orthogonal to the all-ones vector $\mathbf{j}$; otherwise, it is termed *non-main*. The basic results on main eigenvalues can be found in Hagos [16], while a comprehensive survey is given in Rowlinson [20].

A graph G is called a *chain graph* if it is bipartite and the neighborhoods of vertices in each partite set form a chain under inclusion. Equivalently, chain graphs avoid the induced subgraphs $2K_2$, C_3, and C_5. From the perspective of spectral graph theory, they are distinguished among connected bipartite graphs by achieving the maximal spectral radius with respect to both the adjacency and signless Laplacian matrices, for fixed order and size [7]. Chain graphs are also known by other names in the literature, such as *difference graphs* [18], *double nested graphs* [4, 5], and *Ferrers graphs* [15]. Their main eigenvalues have been studied in [1], while some applications can be found in [3]. Additional significant classes of bipartite graphs, namely 4-regular cyclic bipartite graphs and complete $(2, 2)$-bipartite graphs, have also been examined in the literature; see [6] and [17], respectively.

The objective of this paper is to characterize chain graphs that have exactly $n-1$ main eigenvalues. Graphs with this property have already been studied in the literature. It was shown that the unique non-main eigenvalue and its corresponding eigenvector can be derived from the walk matrix without solving the characteristic equation [12]. A new and simple criterion was established for determining when an almost controllable graph is uniquely determined by its generalized spectrum [19]. Furthermore, Du et al. [13] have shown that for almost controllable trees, unicyclic graphs, and bicyclic graphs, the diameter of the complement is at most 3; in addition, all integral almost controllable graphs were characterized. A complete characterization was given for almost controllable cographs, threshold graphs, as well as all almost controllable graphs whose second-largest eigenvalue is at most $\frac{\sqrt{5}-1}{2}$ [14].

The remainder of the paper is organized as follows. Section 2 provides the necessary preliminaries. In Section 3, we present a necessary condition for a chain graph to be almost controllable. Sections 4 and 5 examine the role of certain eigenvalues such as ± 1 or $\frac{1\pm\sqrt{5}}{2}$ and vertex types (downer or neutral) in the structure of such graphs. The main results, including new examples of almost controllable chain graphs, are given in Section 6. Further problems are discussed in Section 7.

2 Preliminaries

In this section, we recall several results from the literature that will be needed in the sequel.

The vertex set of any connected chain graph G can be partitioned into two color classes, which are further subdivided into h non-empty cells as follows:

$$\bigcup_{i=1}^{h} U_i \quad \text{and} \quad \bigcup_{i=1}^{h} V_i.$$

Each vertex in U_s is adjacent (via cross edges) to all vertices in the set $\bigcup_{k=1}^{h+1-s} V_k$, for every $s = 1, 2, \ldots, h$. Let $N_G(w)$ denote the neighborhood of a vertex w in G.

This structure ensures that, for any $u' \in U_{s+1}$ and $u'' \in U_s$, we have $N_G(u') = \cup_{k=1}^{h-s} V_k \subset \cup_{k=1}^{h+1-s} V_k = N_G(u'')$, and similarly, for $v' \in V_{t+1}$ and $v'' \in V_t$, it holds that $N_G(v') \subset N_G(v'')$. This nesting of neighborhoods in both parts of the bipartition illustrates the *double nesting* property, from left to right and from right to left (see Figure 1).

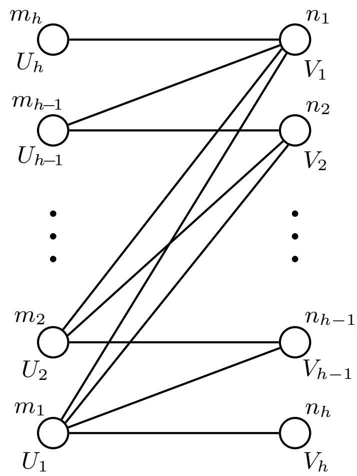

Figure 1: The chain graph $G = \mathrm{DNG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$.

Let $m_s = |U_s|$ and $n_s = |V_s|$ for each $s = 1, 2, \ldots, h$. Then, the graph G can be represented as

$$\mathrm{DNG}(m_1, m_2, \ldots, m_h;\ n_1, n_2, \ldots, n_h),$$

where the notation DNG refers to a *double nested graph*.

Removing a vertex from the graph affects the multiplicity of its eigenvalues, which can either remain unchanged (classifying the vertex as neutral), increase by one (categorizing it as a Parter vertex), or decrease by one, which is associated with downer vertices.

The vertex types in the chain graphs were considered by Anđelić et al. [3]. It was proved that in chain graphs all vertices are either downer or neutral with respect to nonzero eigenvalues (i.e., no Parter vertices are found relative to nonzero eigenvalues). In addition, if v is a neutral vertex for $\lambda \neq 0$, then $\mathbf{x}_v = 0$ in the corresponding eigenvector.

Proposition 2.1. [2, 3] *Let* $G = \mathrm{DNG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$. *Then:*

a) Up to a sign $\phi_G(\lambda) = x^{n-2h} \det M_G(x)$, *where*

$$M_G(x) = \begin{bmatrix} m_1 & x & & & \\ x & n_h & x & & \\ & & \ddots & & \\ & & x & m_h & x \\ & & & x & n_1 \end{bmatrix},$$

b) The spectrum of G *comprises of* 0 *with multiplicity* $n - 2h$ *and of* $2h$ *distinct nonzero eigenvalues;*

c) U_1, U_h, *(resp.* V_1, V_h*) contain only downer vertices for* $\lambda \neq 0$*;*

d) U_i, U_{i+1} *(resp.* V_i, V_{i+1}*) cannot both contain neutral vertices for* $\lambda \neq 0$*;*

e) 0 *is a non-main eigenvalue.*

Two distinct eigenvalues of a graph are called *conjugate* if they are algebraic conjugates; that is, if they satisfy the same minimal polynomial over the field of rational numbers. According to the following proposition, such eigenvalues are either both main or both non-main.

Proposition 2.2. [8, *Proposition 7.8.10*] *Let* μ_i *and* μ_j *be conjugate eigenvalues of a graph* G*. Then* μ_i *is a main eigenvalue of* G *if and only if* μ_j *is a main eigenvalue of* G*.*

In our considerations some results will rely on the characterization of main eigenvalues in a chain graph of the form $\mathrm{DNG}(1, \ldots, 1; 1, \ldots, 1)$.

Theorem 2.1. [1] *Let* $G = \mathrm{DNG}(1, \ldots, 1; 1, \ldots, 1)$ *be a chain graph of order* $2h$*. Then* G *has* h *main and* h *non-main eigenvalues. More precisely,* λ *is main eigenvalue in* G *if and only if* $-\lambda$ *is non-main.*

In addition, we will need the eigenvalues, eigenvectors and the determinant of some special structured tridiagonal matrices.

Theorem 2.2. [10] *Let*

$$T_h = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_h . \tag{1}$$

Then the eigenvalues of T_h *are*

$$\lambda_k = 2 - 2\cos\frac{2k+1}{2h+1}\pi,$$

for $k = 0, \ldots, h-1$ *and the corresponding eigenvectors are of the form*

$$[Q_0(-\lambda_k), Q_1(-\lambda_k), \ldots, Q_{h-1}(-\lambda_k)]^\top,$$

where $Q_0(x) = 1$ *and* $Q_i(x) = U_i(\frac{x}{2}+1) - U_{i-1}(\frac{x}{2}+1)$*. (*$U_i$*,* $i = 0, \ldots, h-1$ *are the Chebyshev polynomials of the second kind,* $U_i(x) = \dfrac{\sin(i+1)\theta}{\sin\theta}$*, for* $x = \cos\theta$*,* $0 < \theta < \pi$*.)*

Lemma 2.1. [11] *Let*

$$A_n = \begin{bmatrix} p & 1 & & & \\ q & p & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & q & p & 1 \\ & & & q & p \end{bmatrix}_n .$$

Then,

$$\det A_n = (\sqrt{q})^n U_n(\frac{p}{2\sqrt{q}}).$$

3 Some Necessary Conditions

The objective of this section is to provide some necessary conditions for a chain graph to be an almost controllable graph, that is, a chain graph of order n possessing exactly $n-1$ main eigenvalues. As noted in the previous section, if $0 \in \sigma(G)$, then 0 is necessarily a non-main eigenvalue. Consequently, we focus on chain graphs of the form

$$\mathrm{DNG}(m_1, \dots, m_h; n_1, \dots, n_h),$$

where all but one of the parameters $m_1, \dots, m_h, n_1, \dots, n_h$ equal 1, and the unique exceptional value is exactly 2. Otherwise, if there exists $m_i > 2$ (resp. $n_j > 2$), or at least two of $m_i, n_j > 1, 1 \leq i, j \leq h$, then by Proposition 2.1 b), e), 0 is a non-main eigenvalue of multiplicity at least 2, which implies that G is not almost controllable. Without loss of generality, suppose that $m_{i_0} = 2$ for some $i_0 \in \{1, \dots, h\}$.

It follows immediately from Theorem 2.1 that, among graphs of the form,

$$G = \mathrm{DNG}(\underbrace{1, \dots, 1}_{h}; \underbrace{1, \dots, 1}_{h}), \quad h \geq 1,$$

only the graph $K_2 = \mathrm{DNG}(1; 1)$ is almost controllable.

For convenience, we denote by $G_h^{i_0}$ the chain graph

$$\mathrm{DNG}(\underbrace{1, \dots, 1, 2, 1, \dots, 1}_{h}; \underbrace{1, \dots, 1}_{h}),$$

where the 2 appears at the i_0-th position of the h-sequence. Similarly, we write H_h to denote the chain graph

$$\mathrm{DNG}(\underbrace{1, \dots, 1}_{h}; \underbrace{1, \dots, 1}_{h}).$$

In the following theorem, we establish a necessary condition for $G_h^{i_0}$ to be almost controllable.

Theorem 3.1. *If $G_h^{i_0}$ is almost controllable, then both vertices in U_{i_0} are downer vertices in G.*

Proof. First, we observe that, by the eigenvalue equations, the entries of any eigenvector for $\lambda \neq 0$, corresponding to the vertices in U_{i_0} must be equal. If $\mathbf{x} = (x_1, \dots, \underbrace{x_{i_0}, x'_{i_0}}_{U_{i_0}}, \dots, x_h, y_1, \dots, y_h)^\top$ is an eigenvector of $G_h^{i_0}$ corresponding to $\lambda \neq 0$, then $\lambda x_{i_0} = \lambda x'_{i_0} = \sum_{i=1}^{h+1-i_0} y_i$, and hence $x_{i_0} = x'_{i_0}$. Suppose, for the sake of contradiction, that G is almost controllable and that there exists a nonzero eigenvalue λ for which a vertex $u_{i_0} \in U_{i_0}$ is neutral in G. Then λ is also an eigenvalue of the induced subgraph

$$G - u_{i_0} = \mathrm{DNG}(1, \dots, 1; 1, \dots, 1).$$

By Theorem 2.1, either λ or $-\lambda$ is a non-main eigenvalue of $G - u_{i_0}$.
Moreover, since the eigenvectors for G and $G - u_{i_0}$ differ only in the coordinate corresponding to u_{i_0}, this implies that λ or $-\lambda$ is also a non-main eigenvalue of G. To be precise, if

$$(x_1, \dots, x_{i_0}, \dots, x_h, y_1, \dots, y_h)^\top,$$

is an eigenvector corresponding to λ (or $-\lambda$) in $G - u_{i_0}$, then

$$(x_1, \ldots, x_{i_0}, 0, x_{i_0+1}, \ldots, x_h, y_1, \ldots, y_h)^\top,$$

is an eigenvector for the same eigenvalue in G. This contradicts the assumption that G is almost controllable as it has at least two non-main eigenvalues, 0 and λ, or 0 and $-\lambda$. □

From the previous theorem, we also deduce that if the vertices in U_{i_0} of $G_h^{i_0}$ are neutral, then $G_h^{i_0}$ cannot be almost controllable. In this scenario, the corresponding vertex in U'_{i_0} of the graph

$$H_h = G_h^{i_0} - u_{i_0}, \quad u_{i_0} \in U_{i_0},$$

is also neutral. Consequently, our subsequent analysis focuses on the occurrence of neutral vertices within H_h.

Lemma 3.1. *Let $G = H_h$. If*

$$(2h + 1)(2\ell + 1) = (2i - 1)(2k + 1),$$

$i \neq 1, h$ for some $\ell \in \{0, 1, \ldots, h - 1\}$, then the vertex $u_i \in U_i$ is neutral for the eigenvalue,

$$\lambda_k = (2 \sin \frac{(2k + 1)\pi}{2(2h + 1)})^{-1}.$$

Proof. The adjacency matrix of G is of the form $\begin{bmatrix} O & N \\ N & O \end{bmatrix}$ for

$$N = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & \\ \vdots & & \iddots & & \\ 1 & 1 & & & \\ 1 & & & & \end{bmatrix},$$

and an eigenvector for an eigenvalue λ is of the form $[\mathbf{x}^\top, \mathbf{x}^\top]^\top$ for $\mathbf{x} = [x_1, \ldots, x_h]^\top$. Then $\mathbf{x}$ is an eigenvector of N^2 for λ^2 and of

$$(N^2)^{-1} = \begin{bmatrix} & & & 1 \\ & & 1 & -1 \\ & \iddots & \iddots & \\ 1 & -1 & & \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} = T_h,$$

for $\frac{1}{\lambda^2}$. According to Theorem 2.2, the eigenvalues of T_h are $\lambda_k = 2 - 2\cos\frac{2k+1}{2h+1}\pi$, for $k = 0, \ldots, h-1$ and the corresponding eigenvector $\mathbf{x}$ can be chosen in the following form

$$\mathbf{x} = [Q_0(-\lambda_k), Q_1(-\lambda_k), \ldots, Q_{h-1}(-\lambda_k)]^\top,$$

where $Q_0(x) = 1$ and $Q_i(x) = U_i(\frac{x}{2} + 1) - U_{i-1}(\frac{x}{2} + 1)$. Therefore, from $\frac{1}{\lambda^2} = 2 - 2\cos(\frac{2k+1}{2h+1}\pi)$, we obtain

$$\lambda = \pm\frac{1}{\sqrt{2 - 2\cos(\frac{2k+1}{2h+1}\pi)}},$$

for some $k = 0, \dots, h-1$. Next,

$$\begin{aligned} Q_{i-1}(-\lambda_k) &= U_{i-1}(-\frac{\lambda_k}{2}+1) - U_{i-2}(-\frac{\lambda_k}{2}+1), \\ &= U_{i-1}(\cos(\frac{2k+1}{2h+1}\pi)) - U_{i-2}(\cos(\frac{2k+1}{2h+1}\pi)), \\ &= \frac{\sin i\theta}{\sin\theta} - \frac{\sin(i-1)\theta}{\sin\theta}, \\ &= 2\frac{\cos(2i-1)\frac{\theta}{2}\sin\frac{\theta}{2}}{\sin\theta}, \end{aligned}$$

for $\theta = \dfrac{2k+1}{2h+1}\pi \in (0,\pi)$. Bearing in mind that

$$\sqrt{2 - 2\cos\frac{2k+1}{2h+1}\pi} = 2\sin\frac{\theta}{2},$$

we obtain

$$Q_{i-1}(-\lambda_k) = \frac{\cos(2i-1)\dfrac{\theta}{2}}{\cos\frac{\theta}{2}},$$

and $\lambda_k = \frac{1}{2\sin\frac{\theta}{2}}$. Then u_i is a neutral vertex in G if and only if $Q_{i-1}(-\lambda_k) = 0$, i.e., if and only if $\cos(2i-1)\dfrac{\theta}{2} = 0$ and $\cos\frac{\theta}{2} \neq 0$. The latter equality occurs if and only if $(2i-1)\frac{\theta}{2} = \frac{\pi}{2} + \ell\pi$ for some integer ℓ, i.e., if and only if $\frac{(2i-1)(2k+1)}{2h+1} = 2\ell+1$. Since $i \in \{1,\dots,h-1\}$, $(i-1 \neq 0, h-1)$ as by Proposition 2.1(c), no neutral vertices are found in U_1 and U_h, for $k \in \{0,1,\dots,h-1\}$. We conclude that $\ell \in \{0,\dots,h-1\}$, and $i \neq 1,h$. Hence, if $(2h+1)(2\ell+1)$ can be factored as $(2i-1)(2k+1)$ with $i \neq 1,h$, then the vertex u_i in U_i is neutral for the eigenvalue $\lambda_k = (2\sin\dfrac{(2k+1)\pi}{2(2h+1)})^{-1}$. This completes the proof. □

Corollary 3.1. *Let $G = G_h^{i_0}$. If $(2h+1)(2\ell+1) = (2i_0-1)(2k+1)$, $i_0 \neq 1,h$, for some $k,\ell \in \{0,1,\dots,h-1\}$, then G is not almost controllable.*

Example 3.1. *Let $h = 10$, so that $\ell \in \{0,1,\dots,9\}$. For example, when $\ell = 0$, using the equality $3\cdot 7 = 21$, we deduce that the vertex in U_2 is neutral for the eigenvalue*

$$\left(2\sin\frac{7\pi}{42}\right)^{-1} = 1.$$

Similarly, the vertex in U_4 is neutral for the eigenvalue

$$\left(2\sin\frac{3\pi}{42}\right)^{-1} \approx 2.24698.$$

Furthermore, from the equalities $3\cdot 21 = 9\cdot 7$ and $5\cdot 21 = 15\cdot 7$, we conclude that vertices in U_5 and U_8 are neutral for the eigenvalue 1. Therefore, the graphs G_{10}^2, G_{10}^4, G_{10}^5, and G_{10}^8 are not almost controllable.

In the sequel we analyze suitable choices for i_0, so that $G_h^{i_0}$ that has only downer vertices in U_{i_0}, is almost controllable.

Lemma 3.2. *Let $G = G_h^{i_0}$. If $i_0 = \lfloor \frac{h}{2} \rfloor + 1$, and the vertices in U_{i_0} are downer for any nonzero eigenvalue, then G is almost controllable.*

Proof. Assume on the contrary, that G is not almost controllable, i.e., that there exists an eigenvalue $\lambda \neq 0$, that is non-main. If

$$\mathbf{x} = (x_1, \dots, x_{i_0}, x_{i_0}, x_{i_0+1}, \dots, x_h, y_1, \dots, y_h)^\top,$$

is the corresponding eigenvector, then

$$x_1 + \cdots + 2x_{i_0} + \cdots + x_h + y_1 + \cdots + y_h = 0, \tag{2}$$

i.e., $\lambda x_1 + \lambda y_1 = 0$ by eigenvalue equations for x_1 and y_1. Since $\lambda \neq 0$, we obtain $x_1 + y_1 = 0$. Next, from

$$\lambda x_h + \lambda y_h = y_1 + x_1 = 0,$$

it follows $x_h + y_h = 0$. We proceed with

$$\lambda x_2 + \lambda y_2 = x_1 + \cdots + 2x_{i_0} + \cdots + x_{h-1} + y_1 + \cdots + y_{h-1} = \lambda x_1 + \lambda y_1 - (x_h + y_h) = 0,$$

to conclude that $x_2 + y_2 = 0$. In a similar way,

$$\lambda x_{h-1} + \lambda y_{h-1} = y_1 + y_2 + x_1 + x_2 = 0.$$

By following the same method we arrive at

$$x_1 + y_1 = x_2 + y_2 = \cdots = x_h + y_h = 0.$$

From (2) we obtain $x_{i_0} = 0$. A contradiction. □

Lemma 3.3. *Let $G = G_h^{i_0}$.*

i) If $i_0 = \lfloor \frac{h}{2} \rfloor$, the vertices in U_{i_0} are downer for any nonzero eigenvalue and $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$ if h is odd (resp. $\pm 1 \notin \sigma(G)$ if h is even), then G is almost controllable.

ii) If $i_0 = \lfloor \frac{h}{2} \rfloor + 2$, the vertices in U_{i_0} are downer for any non-zero eigenvalue and $\pm 1 \notin \sigma(G)$ if h is odd (resp. $\frac{1\pm\sqrt{5}}{2} \notin \sigma(G)$ if h is even), then G is almost controllable.

Proof. i) We first consider the case $h = 2k + 1$. Then $i_0 = k$. As in the proof of Lemma 3.2 we conclude that if $\lambda \neq 0$ is a non-main eigenvalue and $\mathbf{x}$ is the corresponding eigenvector, then

$$x_1 + y_1 = \cdots = x_{i_0} + y_{i_0} = 0,$$

as well as

$$x_{h-i_0+2} + y_{h-i_0+2} = \cdots = x_h + y_h = 0.$$

Then

$$\lambda(x_{h-i_0+1} + y_{h-i_0+1}) = y_1 + \cdots + y_{i_0} + x_1 + \cdots + 2x_{i_0} = x_{i_0}.$$

Hence, we obtain

$$x_{h-i_0+1} + y_{h-i_0+1} = \frac{x_{i_0}}{\lambda}.$$

Then,

$$\begin{aligned}
\lambda(x_{i_0+1} + y_{i_0+1}) &= y_1 + \cdots + y_{h-i_0+2} + x_1 + \cdots + x_{h-i_0+2}, \\
&= \lambda(x_1 + y_1) - (x_h + y_h) - \cdots - (x_{h-i_0+1} + y_{h-i_0+1}), \\
&= -\frac{x_{i_0}}{\lambda},
\end{aligned}$$

and

$$x_{i_0+1} + y_{i_0+1} = -\frac{x_{i_0}}{\lambda^2}.$$

Now, from (2) we obtain

$$x_{i_0} - \frac{x_{i_0}}{\lambda^2} + \frac{x_{i_0}}{\lambda} = 0,$$

i.e.,

$$x_{i_0}(\lambda^2 + \lambda - 1) = 0,$$

which is a contradiction, since $\lambda \neq \frac{-1\pm\sqrt{5}}{2}$ and $x_{i_0} \neq 0$. If h is even, then

$$x_1 + y_1 = \cdots = x_{i_0} + y_{i_0} = x_{h-i_0+2} + y_{h-i_0+2} = \cdots = x_h + y_h = 0,$$

and

$$\lambda(x_{i_0+1} + y_{i_0+1}) = -\frac{x_{i_0}}{\lambda}.$$

Then, (2) implies $x_{i_0} - \frac{x_{i_0}}{\lambda} = 0$, i.e., $x_{i_0} = 0$ or $\pm 1 \in \sigma(G)$. A contradiction. Case ii) is treated in a similar way. For h odd we obtain $x_{i_0} - \frac{x_{i_0}}{\lambda} = 0$, and for h even $x_{i_0} - \frac{x_{i_0}}{\lambda^2} - \frac{x_{i_0}}{\lambda} = 0$, that both lead to a contradiction. $\square$

Remark 3.1. *It is well-known that the spectrum of any bipartite graph is symmetric about* 0*, [9, p.56]. Since* $\pm\frac{-1\pm\sqrt{5}}{2} = \frac{1\pm\sqrt{5}}{2}$*, and* G *is bipartite, we conclude that in i) and ii)* $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$ *and* $\frac{1\pm\sqrt{5}}{2} \notin \sigma(G)$ *are equivalent. Moreover,* $\frac{1+\sqrt{5}}{2}$ *and* $\frac{1-\sqrt{5}}{2}$ *are conjugate numbers over rationals, as they are the roots of* $p(\lambda) = \lambda^2 - \lambda - 1$*. Consequently, if they belong to* $\sigma(G)$*, then they are either both main or non-main eigenvalues of* G*. Similarly, for* $\frac{-1+\sqrt{5}}{2}$ *and* $\frac{-1-\sqrt{5}}{2}$*.*

4 1 and $\dfrac{-1+\sqrt{5}}{2}$ as Eigenvalues of $G_h^{i_0}$

As highlighted in Lemma 3.2, identifying certain almost controllable graphs $G_h^{i_0}$ requires imposing specific restrictions on their spectra. Accordingly, in this section, we derive necessary and sufficient conditions on the parameters h and i_0 that determine whether the eigenvalues ± 1 or $\frac{\pm 1\pm\sqrt{5}}{2}$ appear (or do not appear) in the spectrum of $G_h^{i_0}$.

Lemma 4.1. *Let* $G = G_h^{i_0}$*. Then* 1 *is an eigenvalue of* G *if and only if* $h \equiv 0 \pmod 3$ *and* $i_0 \equiv 0 \pmod 3$ *or* $h \equiv 1 \pmod 3$ *and* $i_0 \equiv 2 \pmod 3$*.*

Proof. A real number λ is an eigenvalue of G if and only if $\phi_G(\lambda) = 0$ i.e., if and only if by Proposition 2.1 a)

$$\varphi_G(\lambda) = \begin{vmatrix} 1 & \lambda & & & & \\ \lambda & 1 & \lambda & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 2 & & \\ & & & \ddots & \ddots & \lambda \\ & & & & \lambda & 1 \end{vmatrix}_{2h} = 0,$$

where 2 appears on the position $2i_0 - 1$ on the main diagonal. Using the linearity of the determinant along the $(2i_0 - 1)$-th column and then expanding along the $(2i_0 - 1)$-th column in the second determinant, we obtain

$$\varphi_G(1) = \begin{vmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{vmatrix}_{2h} + \begin{vmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{vmatrix}_{2i_0-2} \cdot \begin{vmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{vmatrix}_{2h-2i_0+1}.$$

By Lemma 2.1, it follows that

$$\varphi_G(1) = U_{2h}(\frac{1}{2}) + U_{2i_0-2}(\frac{1}{2}) \cdot U_{2h-2i_0+1}(\frac{1}{2}),$$

where the Chebyshev polynomials of the second kind are computed at $\frac{1}{2}$ by

$$U_n(\frac{1}{2}) = \frac{\sin(n+1)\frac{\pi}{3}}{\sin\frac{\pi}{3}},$$

since $\cos\frac{\pi}{3} = \frac{1}{2}$. Taking into account that

$$U_n(\frac{1}{2}) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 6; \\ 0 & \text{if } \quad n \equiv 2 \pmod 3; \\ -1 & \text{if } n \equiv 3, 4 \pmod 6; \end{cases}$$

we analyze nine possible cases for $2h$ and $2i_0 - 2$ modulo 6. As both numbers are even, we only take in consideration $\equiv \pmod 6 \in \{0, 2, 4\}$. If $2h \equiv 0 \pmod 6$ and $2i_0 - 2 \equiv 4 \pmod 6$, then

$$\varphi_G(1) = 1 + (-1) \cdot 1 = 0,$$

as $2h - 2i_0 + 1 \equiv 1 \pmod 6$. Similarly, if $2h \equiv 2 \pmod 6$ and $2i_0 - 2 \equiv 2 \pmod 6$, then $2h - 2i_0 + 1 \equiv 5 \pmod 6$ and

$$\varphi_G(1) = 0 + 0 \cdot 0 = 0.$$

Remaining seven cases are treated in a similar fashion and in all we obtain a nonzero value for $\varphi_G(1)$.

Consequently, $\varphi_G(1) = 0$ if and only if $2h \equiv 0 \pmod 6$ and $2i_0 - 2 \equiv 4 \pmod 6$ or $2h \equiv 2 \pmod 6$ and $2i_0 - 2 \equiv 2 \pmod 6$, i.e., if and only if $h \equiv 0 \pmod 3$ and $i_0 \equiv 0 \pmod 3$ or $h \equiv 1 \pmod 3$ and $i_0 \equiv 2 \pmod 3$. $\square$

Example 4.1. *Let $h = 7$ and $i_0 = 5$. Then*

$$\sigma(G_7^5) = \{\pm 5.00736, \pm 1.80017, \pm 1, \pm 0.826291, \pm 0.636105, \pm 0.564991, \pm 0.52831\},$$

that confirms that $\pm 1 \in \sigma(G_7^5)$.

Lemma 4.2. *Let $G = G_h^{i_0}$. Then $\frac{\pm 1 \pm \sqrt{5}}{2}$ are eigenvalues of G if and only if $h \equiv 1 \pmod 5$ and $i_0 \equiv 0 \pmod 5$ or $h \equiv 2 \pmod 5$ and $i_0 \equiv 3 \pmod 5$.*

Proof. We follow the same procedure as in the proof of Lemma 4.1. If $\lambda = \frac{1+\sqrt{5}}{2} \in \sigma(G)$, then

$$\varphi_G(\frac{1+\sqrt{5}}{2}) = (\frac{1+\sqrt{5}}{2})^{2h} U_{2h}(\frac{1+\sqrt{5}}{2}) + (\frac{1+\sqrt{5}}{2})^{2h-1} U_{2i-2}(\frac{1+\sqrt{5}}{2}) \cdot U_{2h-2i+1}(\frac{1+\sqrt{5}}{2}).$$

The values of Chebyshev polynomials of the second at $\frac{1+\sqrt{5}}{2}$ we compute in the following way:

$$U_n(\frac{1+\sqrt{5}}{2}) = \frac{\sin(n+1)\frac{2\pi}{5}}{\sin\frac{2\pi}{5}} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 5; \\ \frac{\sin(\frac{\pi}{5})}{\sin(\frac{2\pi}{5})} & \text{if } n \equiv 1 \pmod 5; \\ -\frac{\sin(\frac{\pi}{5})}{\sin(\frac{2\pi}{5})} & \text{if } n \equiv 2 \pmod 5; \\ -1 & \text{if } n \equiv 3 \pmod 5; \\ 0 & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

as $\cos\frac{2\pi}{5} = \frac{1}{\sqrt{5}-1}$. We also note that $\frac{\sin(\frac{\pi}{5})}{\sin(\frac{2\pi}{5})} = \frac{-1+\sqrt{5}}{2}$. We consider 25 possible cases with respect to $2h, 2i_0-2 \equiv \pmod 5$ to check when

$$\frac{1+\sqrt{5}}{2}U_{2h}(\frac{1+\sqrt{5}}{2})U_{2i_0-2}(\frac{1+\sqrt{5}}{2})\cdot U_{2h-2i_0+1}(\frac{1+\sqrt{5}}{2}) = 0.$$

If $h \equiv 1 \pmod 5$ and $i_0 \equiv 0 \pmod 5$, then $2h \equiv 2 \pmod 5$ and $2i_0 - 2 \equiv 3 \pmod 5$. Hence,

$$\varphi_G(\frac{1+\sqrt{5}}{2}) = (\frac{1+\sqrt{5}}{2})^{2h-1}\Big(\frac{1+\sqrt{5}}{2}\cdot\frac{1-\sqrt{5}}{2} + (-1)\cdot(-1)\Big) = 0.$$

If $h \equiv 2 \pmod 5$ and $i_0 \equiv 3 \pmod 5$, then $2h \equiv 4 \pmod 5$ and $2i_0-2 \equiv 4 \pmod 5$ and $2h - 2i_0 + 1 \equiv 4 \pmod 5$. Therefore,

$$\varphi_G(\frac{1+\sqrt{5}}{2}) = (\frac{1+\sqrt{5}}{2})^{2h-1}\Big(\frac{1+\sqrt{5}}{2}\cdot 0 + 0\cdot 0\Big) = 0.$$

The remaining 23 cases are dealt with similarly and in all $\varphi_G(\frac{1+\sqrt{5}}{2}) \neq 0$. □

Example 4.2. *Let $h = 7$ and $i_0 = 3$. Then $\frac{\pm1\pm\sqrt{5}}{2} \in \sigma(G_7^3)$. Also, for $h = 6$ and $i_0 = 5$, the corresponding chain graph G_6^5 contains $\frac{\pm1\pm\sqrt{5}}{2}$ in its spectrum.*

5 Downer Vertices in $G_h^{i_0}$ for $i_0 \in \Big\{\lfloor\frac{h}{2}\rfloor, \lfloor\frac{h}{2}\rfloor+1, \lfloor\frac{h}{2}\rfloor+2\Big\}$

According to Lemmas 3.2 and 3.3, to construct an almost controllable graph $G_h^{i_0}$ with

$$i_0 \in \left\{\left\lfloor\frac{h}{2}\right\rfloor, \left\lfloor\frac{h}{2}\right\rfloor+1, \left\lfloor\frac{h}{2}\right\rfloor+2\right\},$$

it is necessary that both vertices in U_{i_0} are downer vertices. In this section, we investigate the conditions under which this occurs.

Lemma 5.1. *Let $G = G_h^{i_0}$ and $i_0 = \lfloor\frac{h}{2}\rfloor + 1$. Then vertices in U_{i_0} are downer for any nonzero eigenvalue.*

Proof. Assume on the contrary that there is $\lambda \neq 0$ such that the vertices in U_{i_0} are neutral for λ. Then, in the eigenvector $\mathbf{x}$ for λ, $x_{i_0} = 0$. Hence, λ is also an eigenvalue of H_h with the corresponding eigenvector of the form $(x_1, \ldots, x_{i_0-1}, 0, x_{i_0+1}, \ldots, x_h, x_1, \ldots, x_h)^\top$. By the eigenvalue equations, we obtain

$$0 = \lambda x_{i_0} = x_1 + \ldots + x_{h-i_0+1} = \begin{cases} x_1 + \ldots + x_{i_0} & \text{if } h \text{ is odd}; \\ x_1 + \ldots + x_{i_0-1} & \text{if } h \text{ is even}. \end{cases}$$

If h is odd, $0 = \lambda x_{i_0} = x_1 + \ldots + x_{i_0-1} + 0 = \lambda x_{i_0+1}$, then $x_{i_0} = x_{i_0+1} = 0$ is a contradiction by Proposition 2.1 d). For h even, $0 = \lambda x_{i_0} = x_1 + \ldots + x_{i_0-1} = x_1 + \ldots + x_{i_0-1} + 0 = x_1 + \cdots + x_{i_0} = \lambda x_{i_0-1}$, we obtain $x_{i_0} = x_{i_0-1} = 0$. A contradiction. □

Lemma 5.2. *Let $G = G_h^{i_0}$.*

i) If $i_0 = \lfloor \frac{h}{2} \rfloor$, and $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$ if h is odd (resp. if $\pm 1 \notin \sigma(G)$ if h is even), then the vertices in U_{i_0} are downer for any nonzero eigenvalue.

ii) If $i_0 = \lfloor \frac{h}{2} \rfloor + 2$, and $\pm 1 \notin \sigma(G)$ if h is odd (resp. $\frac{1\pm\sqrt{5}}{2} \notin \sigma(G)$ if h is even), then the vertices in U_{i_0} are downer for any nonzero eigenvalue.

Proof. i) Assume, on the contrary, that there exists $\lambda \neq 0$, such that the vertices in U_{i_0} are neutral for λ. Then $x_{i_0} = 0$ and $\lambda \in \sigma(H_h)$.
If h is odd, say $h = 2k+1$, then $i_0 = k$ and from the eigenvalue equations we obtain

$$\begin{aligned} \lambda x_{k+2} &= x_1 + \cdots + x_k, \\ \lambda x_{k+1} &= x_1 + \cdots + x_{k+1} = \lambda x_{k+2} + x_{k+1}, \\ 0 = \lambda x_k &= x_1 + \cdots + x_{k+2} = \lambda x_{k+1} + x_{k+2}. \end{aligned}$$

From the last equation it follows that $x_{k+2} = -\lambda x_{k+1}$, that when plugged in the previous one leads to $\lambda x_{k+1} = -\lambda^2 x_{k+1} + x_{k+1}$, i.e., to $(\lambda^2 + \lambda - 1)x_{k+1} = 0$. Hence, we obtain $x_{k+1} = 0$, as $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$. A contradiction, by Proposition 2.1 d).
If $h = 2k$ is even, then $i_0 = k$ and

$$\begin{aligned} \lambda x_{k+1} &= x_1 + \cdots + x_k, \\ 0 = \lambda x_k &= x_1 + \cdots + x_{k+1} = \lambda x_{k+1} + x_{k+1}. \end{aligned}$$

Therefore, $(\lambda + 1)x_{k+1} = 0$, i.e., $x_{k+1} = 0$, as $\pm 1 \notin \sigma(G)$. A contradiction. A proof for ii) is done in a similar way. □

6 Main Results

In this section, we summarize the preceding results with the goal of providing new families of almost controllable graphs.

Theorem 6.1. *Let $G = G_h^{i_0}$ with $i_0 = \lfloor \frac{h}{2} \rfloor + 1$. Then G is almost controllable for any positive integer h.*

Proof. By Lemma 3.2, G is almost controllable provided that the vertices in U_{i_0} are downer for any nonzero eigenvalue. By Lemma 5.1 for any h, the vertices in U_{i_0} are downer. This completes the proof. □

Example 6.1. *If $h = 2k$, then G_{2k}^{k+1} is almost controllable for any positive k and if $h = 2k+1$, then G_{2k+1}^{k+1} is almost controllable for any positive k.*

Theorem 6.2. *Let $G = G_h^{i_0}$ and $i_0 = \lfloor \frac{h}{2} \rfloor$. Then*

i) If h is odd and $h \pmod 5 \in \{0, 3, 4\}$ or $i_0 \pmod 5 \in \{1, 2, 4\}$, then G is almost controllable.

ii) *If h is even and $h \pmod 3 = 2$ or $i_0 \pmod 3 = 1$, then G is almost controllable.*

Proof. By Lemmas 3.3, 5.2, for h odd (resp. h even) G is almost controllable iff $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$ (resp. $\pm 1 \notin \sigma(G)$). Next by Lemma 4.2 if h is odd, $\frac{-1\pm\sqrt{5}}{2} \notin \sigma(G)$ iff

- $h \pmod 5 \in \{0, 2, 3, 4\}$ and $h \pmod 5 \in \{0, 1, 3, 4\}$ or,
- $h \pmod 5 \in \{0, 2, 3, 4\}$ and $i_0 \pmod 5 \in \{0, 1, 2, 4\}$ or,
- $h \pmod 5 \in \{0, 2, 3, 4\}$ and $i_0 \pmod 5 \in \{1, 2, 3, 4\}$ or,
- $i_0 \pmod 5 \in \{0, 1, 2, 4\}$ and $i_0 \pmod 5 \in \{1, 2, 3, 4\}$.

Taking into account that $h = 2i_0 + 1$, we find that the previous reduces to $h \pmod 5 \in \{0, 3, 4\}$ or $i_0 \pmod 5 \in \{1, 2, 4\}$. By Lemma 4.1 we obtain the conclusion for h even. □

Similarly, we obtain,

Theorem 6.3. *Let $G = G_h^{i_0}$ and $i_0 = \lfloor \frac{h}{2} \rfloor + 2$. Then,*

i) *If h is odd and $h \pmod 3 = 2$ or $i_0 \pmod 3 = 1$, then G is almost controllable.*

ii) *If h is even and $h \pmod 5 \in \{0, 3, 4\}$ or $i_0 \pmod 5 \in \{1, 2, 4\}$, then G is almost controllable.*

7 Discussion

Almost controllable chain graphs contain exactly one of the $2h$ vertex partitions with two vertices (the remaining partitions are singletons). In several cases we were able to determine the positions of the corresponding vertices. In a preliminary version of this manuscript we conjectured the following:

> *A chain graph $G = G_h^{i_0}$ is almost controllable if and only if the vertices in U_{i_0} are downer for every nonzero eigenvalue.*

Following a referee's suggestion to illustrate the validity of the conjecture on chain graphs of small order, we examined such examples and found a counterexample that does not satisfy the above conjecture.

Example 7.1. *Let $G = G_3^3$, i.e., $G = \mathrm{DNG}(1, 1, 2; 1, 1, 1)$. Then $-1 \in \sigma(G)$ and a corresponding eigenvector is*

$$(1, 0, -1, -1, -1, 1, 1)^\top.$$

It is easily verified that -1 is a non-main eigenvalue and that the entries corresponding to U_3 are nonzero, and hence downers.

However, the following problems remain open for further research.

- Problem 1. Fully characterize all chain graphs that are almost controllable.
- Problem 2. Identify classes of chain graphs with $n - 2$ main eigenvalues.

Conflicts of Interest The authors declare no conflict of interest.

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